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Imaginary-Time Path Integrals for Three Magnetic Relativistic Schrödinger Operators*

By

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Abstract

After brief introduction to path integral, we consider the problem with three magnetic relativistic Schrödinger operators corresponding to the classical relativistic Hamiltonian symbol with magnetic vector and electric scalar potentials. We discuss their difference in general and their coincidence in the case of constant magnetic fields, as well as whether they are covariant under gauge transformation. Then results are surveyed on path integral representations for their respective imaginary-time relativistic Schrödinger equations, i.e. heat equations, by means of the probability path space measure coming from the Lévy process concerned.

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§ 1. Introduction

In these notes, we consider the quantum Hamiltonian corresponding to the classical relativistic Hamiltonian symbol

$$(1.1) \quad \sqrt{(\xi - A(x))^2 + m^2} + V(x), \quad (\xi, x) \in \mathbf{R}^d \times \mathbf{R}^d$$

for a *spinless* particle with mass m , which is the sum of the kinetic energy term involving magnetic vector potential $A(x)$ and the potential energy term of electric scalar potential $V(x)$. There are in the literature three kinds of quantum relativistic Hamiltonians depending on how to quantize the kinetic energy term $\sqrt{(\xi - A(x))^2 + m^2}$. We call them the *relativistic Schrödinger operators*. We observe their difference in general, and next discuss their coincidence when the vector potential $A(x)$ is linear in x , in particular, in the case of constant magnetic fields, as well as handle whether they are gauge-covariant. Then, on this occasion, we would like to make survey, which might be of some interest, on the results on path integral representations for their respective imaginary-time unitary groups, i.e. real-time semigroups, by means of the probability path space measure coming from the Lévy process concerned. It will be of some interest to collect them in one place to observe how they look like and different, though all the three are essentially connected with the Lévy process. Finally, an anecdote is referred to between Feynman and Dirac concerning the subject.

We know that the authentic operator in relativistic quantum mechanics is the Dirac operator for a *spinning* particle with mass m , which is the first-order system of partial differential operators corresponding to the symbol $\sum_{j=1}^3 \alpha_j(\xi_j - A_j(x)) + m\alpha_4 + V(x)$, where $\alpha := (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ are the four 4×4 -Dirac matrices. The magnetic relativistic Schrödinger operator without scalar potential ($V = 0$) is considered to be the *positive* kinetic energy part of the Dirac operator. For the path integral for the Dirac equation in space-dimension $d = 1$ and in *real time*, i.e. in the Minkowski space-time of *two* dimensions, we refer to [I82], [I84], [ITa84], [ITa88], [ITa87] with its survey [I93], and [BCSS85], [CSS86], [Z88], [Z89].

The description of these notes is of expository character, beginning with a brief introduction to Feynman path integral.

§ 2. Brief Introduction to Path Integral

§ 2.1. What is path integral?

It is a fabulous technique invented by Richard P. Feynman in his Princeton 1942 thesis (see [F05]) and his 1948 paper [F48] to give alternative formulation of quantum mechanics. Its like has never been made before or since. In fact, because of universality of its idea it has now come to prevail over all the domains in quantum physics. He came to the idea, though, as in [F48]

he himself wrote, “suggested by some of Dirac’s remarks ([D33], [D35], [D45]) concerning the relation of classical action to quantum mechanics.” It is a special kind of *functional integral* like

$$\int e^{(i/\hbar)S(X)} \mathcal{D}[X]$$

on space of paths $X: [0, t] \ni s \mapsto X(s) \in \mathbf{R}^d$ with respect to a ‘measure’ $\mathcal{D}[X]$ on the space of these paths, where $S(X)$ is integral of the Lagrangian $L(X)$, $S(X) = \int_0^t L(X(s))ds$, called *action*.

Consider the nonrelativistic Schrödinger equation for one particle with mass m :

$$(2.1) \quad i\hbar \frac{\partial}{\partial t} \psi(t, x) = \left[-\frac{\hbar^2}{2m} \Delta + V(x) \right] \psi(t, x), \quad t > s, \quad x \in \mathbf{R}^d,$$

where $\hbar = h/(2\pi)$ ($h > 0$: Planck’s constant). The solution is expressed as

$$\psi(t, x) = \int K(t, x; s, y) f(y) dy$$

with integral kernel $K(t, x; s, y)$, called *fundamental solution* or *propagator*.

Feynman writes down this important quantity $K(t, x; s, y)$ as an ‘integral’

$$(2.2) \quad K(t, x; s, y) = \int_{\{X: X(s)=y, X(t)=x\}} e^{iS(X)/\hbar} \mathcal{D}[X],$$

where $S(X)$ in our present case is given by

$$(2.3) \quad S(X) = \int_s^t \left[\frac{m}{2} \dot{X}(\tau)^2 - V(X(\tau)) \right] d\tau, \quad \dot{X}(\tau) = \frac{d}{d\tau} X(\tau).$$

Here $\mathcal{D}[X]$ stands for a uniform ‘measure’, if it exists, on the space of paths $X(\cdot)$ starting from position y at time s to arrive at position x at time t , *formally*, to be given by the product of continuously-many numbers of the Lebesgue measures $dX(\tau)$ on \mathbf{R}^d for each individual τ :

$$\mathcal{D}[X] := \text{“constant”} \times \prod_{s \leq \tau \leq t} dX(\tau),$$

where the “constant” should be something like $\prod_{s \leq \tau \leq t} \frac{m^{1/2}}{(2\pi i \hbar d\tau)^{1/2}}$, if one dares to try to write it, wondering what it means at all. The right-hand side of (2.2) is what is called *Feynman path integral* or, nowadays simply, *path integral*.

To explain this, Feynman put the following Two Postulates which turn out to be equivalent to get the above expression (2.2) for $K(t, x; s, y)$.

(i) $K(t, x; s, y)$ is the total probability amplitude for the event that the particle starts from position y at time s and arrives at position x at time t . If $\varphi[X]$ stands for the probability amplitude

for the event that it does this motion *along each individual path* $X(\cdot)$, $K(t, x; s, y)$ is the sum of the $\varphi[X]$ over all these paths $X(\cdot)$:

$$(2.4) \quad K(t, x; s, y) = \sum_{X: X(s)=y, X(t)=x} \varphi[X].$$

(ii) The contribution $\varphi[X]$ from each $X(\cdot)$ to the total probability amplitude $K(t, x; s, y)$ is given by

$$(2.5) \quad \varphi[X] = C e^{iS(X)/\hbar},$$

where C is a constant independent of path $X(\cdot)$.

These two postulates can be paraphrased:

In quantum mechanics there rules Principle of Democracy that each individual path $X(\cdot)$ contributes to the total probability amplitude $K(t, x; s, y)$ with equal *weight* (*absolute value* in mathematics) and its personanlity is expressed by its *phase* (*argument* in mathematics).

In this respect, in classical mechanics there does not rule Principle of Democracy, because the particle takes the *particular path* between two space-time points (s, y) and (t, x) which makes the action $S(X)$ stationary, called *classical trajectory*. It is the path determined by Euler-Lagrange equation or, in the present case, Newton's equation of motion: $m\ddot{X}(\tau) = -\nabla V(X(\tau))$.

The most characteristic feature of these postulates lies in equation (2.5), which says that the amplitude $\varphi[X]$ is propotional to the phase $e^{iS(X)/\hbar}$. The phrase "propotional to" is that which Feynman determined to substitute for what Dirac had meant by the phrase "analogous to" in [D33, D35], [D45] far before Feynman, by showing after his own analysis and deliberation that *indeed this exponential function could be used in this manner directly* (see Preface of [FH65]). In §6 we shall come back to this subjecte again.

§ 2.2. How to Make It Mathematics?

Here we refer, among others, only to two methods; one is by finite-dimensional approximation, and the other by imaginary-time path integral. In fact, it is by the first method that Feynman himself confirmed his idea of path integral. He calculated $K(t, x; s, y)$ by time-sliced approximation, making partition of the time interval $[s, t]$: $s = t_0 < t_1 < \dots < t_n = t$, $(t_k - t_{k-1} = t/n)$, $x_k := X(t_k)$, $x_0 = X(0) = y$, $x_n = X(t) = x$, as the limit of

$$K_n(t, x; s, y) := \frac{\int_{(\mathbb{R}^d)^{n-1}} \exp \left[\frac{it}{\hbar n} \sum_{k=0}^{n-1} \left(\frac{1}{2} \left(\frac{x_{k+1} - x_k}{t/n} \right)^2 - V(x_k) \right) \right] dx_1 \cdots dx_{n-1}}{\int_{(\mathbb{R}^d)^{n-1}} \exp \left[\frac{it}{\hbar n} \sum_{k=0}^{n-1} \frac{1}{2} \left(\frac{x_{k+1} - x_k}{t/n} \right)^2 \right] dx_1 \cdots dx_{n-1}}$$

as $n \rightarrow \infty$, to ascertain it to satisfy the Schrödinger equation (2.1).

Now we come to the second method, which the present note will be mainly concerned with. First note with (2.2) that the solution of the Schrödinger equation (1.1) turns out to be given by

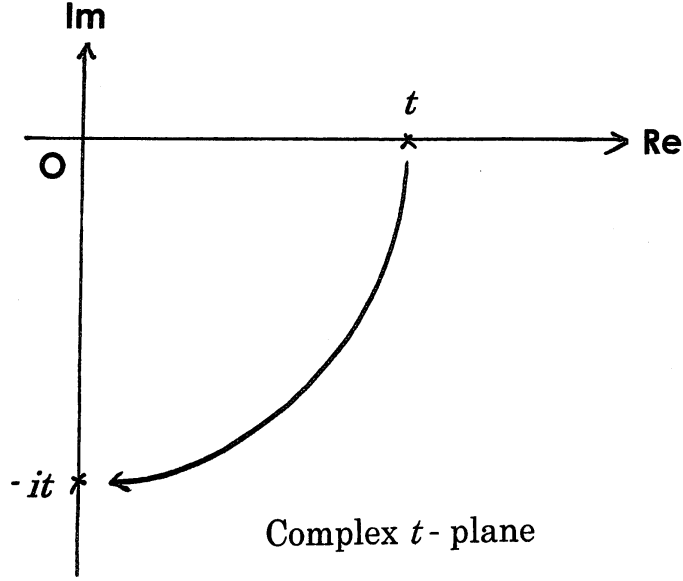


Figure 1. From real time t to imaginary time $-it$

a heuristic path integral

$$(2.6) \quad \psi(t, x) = \int_{\mathbf{R}^d} K(t, x; s, y) \psi(s, y) dy = \int_{\{X: X(t)=x\}} e^{iS(X)/\hbar} \psi(s, X(s)) \mathcal{D}[X].$$

Next, one should know that $\mathcal{D}[X]$ itself does not in general exist in this situation as a countably additive measure. Therefore we cannot go further. But if we rotate in complex t -plane by -90° : $t \rightarrow -it$ (real time t to imaginary time $-it$), i.e. if we go from our Minkowski space-time to *Euclidian* space-time (see Figure 1), the situation changes. Before actually doing it, for simplify put $\hbar = 1$ and $s = 0$. Then our (real-time) Schrödinger equation (2.1) goes to the imaginary-time Schrödinger equation, i.e. heat equation [formally putting $u(t, x) := \psi(-it, x)$]

$$(2.7) \quad \frac{\partial}{\partial t} u(t, x) = \left[\frac{1}{2m} \Delta - V(x) \right] u(t, x), \quad t > 0, \quad x \in \mathbf{R}^d.$$

Simultaneously, the action $S(X)$ in (2.3) changes to integral of the Hamiltonian, and so $iS(X)$ to time integral of the Hamiltonian $-\int_0^t \left[\frac{1}{2m} \dot{X}(\tau)^2 + V(X(\tau)) \right] d\tau$. Then $K(t, x; 0, y)$ changes to

$$(2.8) \quad K^E(t, x; 0, y) = \int_{\{X: X(0)=y, X(t)=x\}} e^{-\int_0^t \left[\frac{1}{2m} \dot{X}(\tau)^2 + V(X(\tau)) \right] d\tau} \mathcal{D}[X],$$

where the superscript “ E ” is attributed to “Euclidean”, and $K^E(t, x; 0, y)$ should become the heat kernel for equation (2.6). However, so as to be able to reach the so-called *Feynman-Kac*

formula, we put here $X_0(\tau) := X(t - \tau)$, $0 \leq \tau \leq t$, to transform paths $X(\cdot)$ to paths $X_0(\cdot)$ and then get from (2.8)

$$(2.9) \quad K^E(t, x; 0, y) = \int_{\{X: X_0(0)=x, X_0(t)=y\}} e^{-\int_0^t [\frac{1}{2m} \dot{X}_0(\tau)^2 + V(X_0(\tau))] d\tau} \mathcal{D}[X_0],$$

so that the solution of (2.6) should be given by the following path integral

$$(2.10) \quad u(t, x) = \int_{\mathbf{R}^d} K^E(t, x; 0, y) g(y) dy = \int_{\{X: X_0(0)=x\}} e^{-\int_0^t [\frac{1}{2m} \dot{X}_0(\tau)^2 + V(X_0(\tau))] d\tau} g(X_0(t)) \mathcal{D}[X_0].$$

Remarkable is that N. Wiener, already around 1923, had constructed a countably additive measure $\mu_x(X_0)$, for each $x \in \mathbf{R}^d$, on the space $C_x := C_x([0, \infty) \rightarrow \mathbf{R}^d)$ of the continuous paths (*Brownian motions*) $B: [0, \infty) \rightarrow \mathbf{R}^d$ starting from $B(0) = x$ at time $t = 0$. This $\mu_x(\cdot)$ is called *Wiener measure*, which is a probability measure on C_0 with characteristic function

$$\exp\left[-t \frac{\xi^2}{2m}\right] = \int_{C_x([0, \infty) \rightarrow \mathbf{R}^d)} e^{iB(t)\xi} d\mu_x(B).$$

Around 1947, Mark Kac, who had been at Cornell University as Feynman and seemed to have heard his lecture, used the Wiener measure to represent the solution $u(t, x)$ of the Cauchy problem of the heat equation (2.6) with initial data $u(0, x) = f(x)$ as a *genuine functional integral*

$$(2.11) \quad u(t, x) = \int K^E(t, x; 0, y) f(y) dy = \int_{C_x([0, \infty) \rightarrow \mathbf{R}^d)} e^{-\int_0^t V(B(s)) ds} f(B(t)) d\mu_x(B).$$

This is the *Feynman-Kac formula* [K66,80] already mentioned above. Thus, identify the path $X_0(\cdot)$ appearing on the right-hand side of (2.9)/(2.10) with the continuous path $B(\cdot)$ in C_x , then the Wiener measure $\mu_x(\cdot)$ turns out to be constructed from the factor “ $e^{-\int_0^t \frac{1}{2m} \dot{B}(\tau)^2 d\tau} \mathcal{D}[B]$ ” on the right-hand side of (2.9)/(2.10).

§ 3. Three Magnetic Relativistic Schrödinger Operators

We consider the quantized operator $H := H_A + V$ corresponding to the classical Hamiltonian

$$(3.1) \quad \sqrt{(\xi - A(x))^2 + m^2} + V(x), \quad (\xi, x) \in \mathbf{R}^d \times \mathbf{R}^d,$$

for a *relativistic* particle of mass m under *magnetic* vector potential $A(x)$ and electric scalar potential $V(x)$. This H is used for a spinless particle in electromagnetic fields in the situation where we may ignore quantum-field theoretic effect like particles creation and annihilation but should take relativistic effect into consideration.

In this note, we pay attention to the following three quantized operators $H^{(1)}$, $H^{(2)}$ and $H^{(3)}$ corresponding to the classical relativistic Hamiltonian symbol (3.1). Their difference is in how to define the first term on the right, H_A , corresponding to the symbol $\sqrt{(\xi - A(x))^2 + m^2}$.

For simplicity, it is assumed here and throughout this note that $A(x)$ is smooth and $V(x)$ is bounded below.

Definition 3.1. The first $H^{(1)} := H_A^{(1)} + V$ is defined with the first term on the right $H_A^{(1)}$ being the Weyl pseudo-differential operator through *mid-point* prescription (e.g. [ITa86, I89, I95]):

$$(3.2) \quad \begin{aligned} (H_A^{(1)} f)(x) &:= \frac{1}{(2\pi)^d} \iint_{\mathbf{R}^d \times \mathbf{R}^d} e^{i(x-y) \cdot \xi} \sqrt{\left(\xi - A\left(\frac{x+y}{2}\right)\right)^2 + m^2} f(y) dy d\xi \\ &= \frac{1}{(2\pi)^d} \iint_{\mathbf{R}^d \times \mathbf{R}^d} e^{i(x-y) \cdot (\xi + A(\frac{x+y}{2}))} \sqrt{\xi^2 + m^2} f(y) dy d\xi \end{aligned}$$

Definition 3.2. The second $H^{(2)} := H_A^{(2)} + V$ is defined with term $H_A^{(2)}$ being the pseudo-differential operator modified by Iftimie-Măntoiu-Purice [IfMP07], [IfMP08], [IfMP10]:

$$(3.3) \quad (H_A^{(2)} f)(x) := \frac{1}{(2\pi)^d} \iint_{\mathbf{R}^d \times \mathbf{R}^d} e^{i(x-y) \cdot (\xi + \int_0^1 A((1-\theta)x + \theta y) d\theta)} \sqrt{\xi^2 + m^2} f(y) dy d\xi.$$

Here the integrals in (3.2), (3.3) on the right-hand side are oscillatory integrals with f being a function in $C_0^\infty(\mathbf{R}^d)$ or in $\mathcal{S}(\mathbf{R}^d)$.

Definition 3.3. The third $H^{(3)} := H_A^{(3)} + V$ is defined with term $H_A^{(3)}$ being the square root of the nonnegative selfadjoint operator $(-i\nabla - A(x))^2 + m^2$:

$$(3.4) \quad H_A^{(3)} := \sqrt{(-i\nabla - A(x))^2 + m^2} + V(x).$$

This $H_A^{(3)}$ does not seem to be defined as a pseudo-differential operator corresponding to a certain tractable symbol. So long as it is defined through Fourier and inverse-Fourier transforms, the candidate of its symbol will not be $\sqrt{(\xi - A(x))^2 + m^2}$.

The last $H^{(3)}$ is used, for instance, to study “stability of matter” in relativistic quantum mechanics in E. Lieb and R. Seiringer [LSei10].

Needless to say, we can show these three relativistic Schrödinger operators $H^{(1)}$, $H^{(2)}$ and $H^{(3)}$ define *selfadjoint* operators in $L^2(\mathbf{R}^d)$, which are bounded from below and, in general, different from one another. In fact further, the three magnetic relativistic Schrödinger operators $H_A^{(1)}$, $H_A^{(2)}$ and $H_A^{(3)}$ are bounded from below by the *same lower bound* m . This was shown for $H_A^{(1)}$ in [I89] with the aid of its expression (5.5) in §5 instead of (3.2) and similarly can be for $H_A^{(2)}$ with the aid of (5.15) instead of (3.3), while it is trivial for $H_A^{(3)}$.

§ 4. Gauge Covariance for Magnetic Relativistic Schrödinger Operators

Among these three magnetic relativistic Schrödinger operators $H_A^{(1)}$, $H_A^{(2)}$ and $H_A^{(3)}$, the Weyl quantized one like $H_A^{(1)}$ (in general, the Weyl pseudo-differential operator) is *compatible well with path integral*. But it is pity that, for general vector potential $A(x)$ it is in general not covariant under gauge transformation, namely, there exists a real-valued function $\varphi(x)$ for which it fails to hold that $H_{A+\nabla\varphi}^{(1)} = e^{i\varphi} H_A^{(1)} e^{-i\varphi}$.

However, $H_A^{(2)}$ (and so $H^{(2)}$) and $H_A^{(3)}$ (and so $H^{(3)}$) are gauge-covariant, though these three are not equal in general. Let us observe these facts in the following.

First, why $H_A^{(3)} = \sqrt{(-i\nabla - A(x))^2 + m^2}$ is gauge-covariant is because the selfadjoint operator $(-i\nabla - A(x))^2 + m^2$ inside $\sqrt{\cdots}$ is gauge-covariant. Next, as in the following proposition, it is easy to show that the modified $H_A^{(2)}$ is gauge-covariant. This property was emphasized in [IfMP07], [IfMP08], [IfMP10] in contrast to $H_A^{(1)}$.

Proposition 4.1. $H_A^{(2)}$ is covariant under gauge transformation, i.e. it follows for any $\varphi \in S(\mathbf{R}^d)$ that $H_{A+\nabla\varphi}^{(2)} = e^{i\varphi} H_A^{(2)} e^{-i\varphi}$. Therefore, so is $H^{(2)}$.

The proof is due to the mean value theorem.

Theorem 4.2. If $A(x)$ is linear in x , i.e. if $A(x) = \dot{A} \cdot x$ with \dot{A} being any $d \times d$ real symmetric constant matrix, then $H_A^{(1)}$, $H_A^{(2)}$ and $H_A^{(3)}$ coincide. In particular, this holds for uniform magnetic fields for $d = 3$.

Proof is omitted.

§ 5. Imaginary-Time Path Integrals for Magnetic Relativistic Schrödinger Operators

Now, let H be one of the magnetic relativistic Schrödinger operators $H^{(1)}$, $H^{(2)}$, $H^{(3)}$ in Definitions 3.1, 3.2, 3.3. In the same way as in the nonrelativistic case, start from (real-time) relativistic Schrödinger equation $i \frac{\partial}{\partial t} \psi(t, x) = H \psi(t, x)$. Rotate it by -90° from real time t to imaginary time $-it$ in complex t -plane, we arrive at the imaginary-time relativistic Schrödinger equation, “heat equation” for $H - m$ [formally putting $u(t, x) := \psi(-it, x)$]:

$$(5.1) \quad \begin{cases} \frac{\partial}{\partial t} u(t, x) = -[H - m]u(t, x), & t > 0, \\ u(0, x) = g(x), & x \in \mathbf{R}^d. \end{cases}$$

The semigroup $u(t, x) = (e^{-t[H-m]}g)(x)$ gives the solution of this Cauchy problem. We want to deal with path integral representation for each $e^{-[H^{(j)}-m]t}g$ ($j = 1, 2, 3$). The relevant path integral is connected with the *Lévy process* [IkW81,89; Ap09] on the space $D_x := D_x([0, \infty) \rightarrow \mathbf{R}^d)$ of the “*càdlàg* paths”, i.e. right-continuous paths $X: [0, \infty) \rightarrow \mathbf{R}^d$ having left-hand limits, and with $X(0) = x$. The associated path space measure is a probability measure λ_x , for each $x \in \mathbf{R}^d$, on $D_x([0, \infty) \rightarrow \mathbf{R}^d)$ whose characteristic function is given by

$$(5.2) \quad e^{-t[\sqrt{\xi^2 + m^2} - m]} = \int_{D_x([0, \infty) \rightarrow \mathbf{R}^d)} e^{i(X(t)-x) \cdot \xi} d\lambda_x(X), \quad t \geq 0, \quad \xi \in \mathbf{R}^d.$$

We are going to start on task of representing the semigroup $e^{-t[H-m]}g$ by path integral. But before that, we want briefly touch on what a kind of path integral expression emerges

heuristically in the present relativistic case (cf.[I93, pp. 26–29, § 5]), to compare it with the nonrelativistic case (2.9)/(2.10), which together with (2.2)/(2.6) is called *configuration space path integral*. Though taking the same procedure as before to find it, we turn out to learn it to be given by *phase space path integral*.

However, to see it, as the general case for H is dependent on which of the three relativistic Schrödinger operators is dealt with, so we do only with the case $H_0 := \sqrt{-\Delta + m^2} + V(x)$ without vector potential $A(x)$. Then we have for the solution of (5.1) with H_0 in place of H

$$(5.3) \quad \begin{aligned} u(t, x) &= (e^{-t[H_0 - m]}g)(x) \\ &= \iint_{\{(P, X); \text{arbitrary}, X(0)=x\}} e^{-\int_0^t \{iP(s)dX(s) + [\sqrt{P(s)^2 + m^2} - m + V(X(s))]\}ds} g(X(t)) \mathcal{D}[P] \mathcal{D}[X]. \end{aligned}$$

Here $\mathcal{D}[P] \mathcal{D}[X] := \prod_{0 \leq \tau \leq t} \frac{dP(\tau) dX(\tau)}{(2\pi)^d}$ is a ‘measure’ on the space of the phase space paths (i.e. momentum and position paths) $(P, X): [0, t] \ni s \mapsto (P(s), X(s)) \in \mathbf{R}^d \times \mathbf{R}^d$ with $X(0) = x$ and, for each fixed τ , $dP(\tau) dX(\tau)$ is the Lebesgue measure on $\mathbf{R}^{2d} = \mathbf{R}^d \times \mathbf{R}^d$. It will turn out that the measure $\lambda_x(\cdot)$ is to be constructed from the factor

$$\left(\int_{\{P: \text{arbitrary}\}} e^{-\int_0^t \{iP(s)dX(s) + [\sqrt{P(s)^2 + m^2} - m]\}ds} \mathcal{D}[P] \right) \mathcal{D}[X]$$

on the right-hand side of (5.3), so that we have a correct *functional integral representaion* quite similar to the nonrelativistic case (2.11):

$$(5.4) \quad u(t, x) = (e^{-t[H_0 - m]}g)(x) = \int_{D_x([0, \infty) \rightarrow \mathbf{R}^d)} e^{-\int_0^t V(X(s))ds} g(X(t)) d\lambda(X).$$

Now we turn to come to the situation involving also the vector potential $A(x)$.

(1) First consider the case for the Weyl pseudo-differential operator $H^{(1)} = H_A^{(1)} + V$ in Definition 3.1. The part $H_A^{(1)}$ can be rewritten as the integral operator:

$$(5.5) \quad \begin{aligned} ([H_A^{(1)} - m]f)(x) &= - \int_{|y|>0} [e^{-iy \cdot A(x + \frac{y}{2})} f(x + y) - f(x) - I_{\{|y|<1\}} y \cdot (\nabla - iA(x)) f(x)] n(dy) \\ &= - \lim_{r \downarrow 0} \int_{|y| \geq r} [e^{-iy \cdot A(x + \frac{y}{2})} f(x + y) - f(x)] n(dy) \\ &= - \text{p.v.} \int_{|y|>0} [e^{-iy \cdot A(x + \frac{y}{2})} f(x + y) - f(x)] n(dy). \end{aligned}$$

Here $n(dy) = n(y)dy$ is an m -dependent measure on $\mathbf{R}^d \setminus \{0\}$, called *Lévy measure* with density

$$(5.6) \quad n(y) = \begin{cases} 2 \left(\frac{m}{2\pi} \right)^{(d+1)/2} \frac{K_{(d+1)/2}(m|y|)}{|y|^{(d+1)/2}}, & m > 0, \\ \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}} \frac{1}{|y|^{d+1}}, & m = 0. \end{cases}$$

$n(dy)$ appears in the Lévy-Khinchin formula:

$$(5.7) \quad \sqrt{\xi^2 + m^2} - m = - \int_{|y|>0} [e^{iy \cdot \xi} - 1 - i\xi \cdot y I_{\{|y|<1\}}] n(dy) = - \lim_{r \rightarrow 0+} \int_{|z| \geq r} [e^{iz \cdot \xi} - 1] n(dz).$$

Proof of (5.5). By the Lévy-Khinchin formula (5.7),

$$\begin{aligned} (H_A^{(1)} f)(x) &= (2\pi)^{-d} \iint e^{i(x-y) \cdot (\xi + A(\frac{x+y}{2}))} \left(m - \lim_{r \rightarrow 0+} \int_{|z| \geq r} [e^{iz \cdot \xi} - 1] n(dz) \right) f(y) dy d\xi \\ &= (2\pi)^{-d} \left[m \iint e^{i(x-y) \cdot \xi} e^{i(x-y) \cdot A(\frac{x+y}{2})} dy d\xi \right. \\ &\quad \left. - \lim_{r \rightarrow 0+} \iiint_{|z| \geq r} (e^{i(x-y+z) \cdot \xi} - e^{i(x-y) \cdot \xi}) n(dz) e^{i(x-y) \cdot A(\frac{x+y}{2})} f(y) dy d\xi \right] \\ &= m \int \delta(x-y) e^{i(x-y) \cdot A(\frac{x+y}{2})} f(y) dy \\ &\quad - \lim_{r \rightarrow 0+} \iint_{|z| \geq r} (\delta(x-y+z) - \delta(x-y)) n(dz) e^{i(x-y) \cdot A(\frac{x+y}{2})} f(y) dy \\ &= m f(x) - \lim_{r \rightarrow 0+} \iint_{|z| \geq r} (e^{-iz \cdot A(x+\frac{z}{2})} f(x+z) - f(x)) n(dz). \end{aligned}$$

□

To represent $e^{-t[H^{(1)}-m]}g$ by path integral, we need some further notations from Lévy process.

For each path $X(\cdot)$, $N_X(dsdy)$ denotes the *counting measure* on $[0, \infty) \times (\mathbf{R}^d \setminus \{0\})$ to count the number of discontinuities of $X(\cdot)$, i.e.

$$(5.8) \quad N_X((t, t'] \times U) := \#\{s \in (t, t']; 0 \neq X(s) - X(s-) \in U\},$$

where $0 < t < t'$ and $U \subset \mathbf{R}^d \setminus \{0\}$ is a Borel set. It satisfies $\int_{D_x} N_X(dsdy) d\lambda_x(X) = ds n(dy)$. Put $\tilde{N}_X(dsdy) := N_X(dsdy) - ds n(dy)$, which may be thought of as a renormalization of $N_X(dsdy)$. Then any path $X \in D_x([0, \infty) \rightarrow \mathbf{R}^d)$ can be expressed with $N_X(\cdot)$ and $\tilde{N}_X(\cdot)$ as

$$(5.9) \quad X(t) = x + \int_0^{t+} \int_{|y| \geq 1} y N_X(dsdy) + \int_0^{t+} \int_{0 < |y| < 1} y \tilde{N}_X(dsdy).$$

Now we have the following path integral representation for $e^{-t[H^{(1)}-m]}g$.

Theorem 5.1 ([ITa86], [I95]).

$$(e^{-t[H^{(1)}-m]}g)(x) = \int_{D_x([0, \infty) \rightarrow \mathbf{R}^d)} e^{-S^{(1)}(t, X)} g(X(t)) d\lambda_x(X),$$

$$\begin{aligned}
S^{(1)}(t, X) &= i \int_0^{t+} \int_{|y| \geq 1} A(X(s-) + \frac{y}{2}) \cdot y N_X(ds dy) + i \int_0^{t+} \int_{0 < |y| < 1} A(X(s-) + \frac{y}{2}) \cdot y \tilde{N}_X(ds dy) \\
(5.10) \quad &+ i \int_0^t ds \text{ p.v. } \int_{0 < |y| < 1} A(X(s) + \frac{y}{2}) \cdot y n(dy) + \int_0^t V(X(s)) ds.
\end{aligned}$$

Proof. We only give a sketch. Put

$$(5.11) \quad (T(t)g)(x) := \int_{\mathbf{R}^d} k_0(t, x-y) e^{-iA\left(\frac{x+y}{2}\right) \cdot (y-x) - V\left(\frac{x+y}{2}\right)t} g(y) dy,$$

where $k_0(t, x-y)$ is the integral kernel of $e^{-t(\sqrt{-\Delta+m^2}-m)}$. Then we can rewrite it as

$$(T(t)g)(x) = \int_{D_x} e^{-iA\left(\frac{x+X(t)}{2}\right) \cdot (X(t)-x) - V\left(\frac{x+X(t)}{2}\right)t} g(X(t)) d\lambda_x(X)$$

with partition of $[0, t]$: $0 = t_0 < t_1 < \dots < t_n = t$, $t_j - t_{j-1} = t/n$,

$$(5.12) \quad S_n(x_0, \dots, x_n) := i \sum_{j=1}^n A\left(\frac{x_{j-1} + x_j}{2}\right) \cdot (x_j - x_{j-1}) + \sum_{j=1}^n V\left(\frac{x_{j-1} + x_j}{2}\right) \frac{t}{n},$$

where $x_j = X(t_j)$ ($j = 0, 1, 2, \dots, n$); $x = x_0 = X(t_0)$, $y = x_n = X(t_n) \equiv X(t)$.

Substitute these $n+1$ points of path $x_j = X(t_j)$ into $S_n(x_0, \dots, x_n)$ to get

$$\begin{aligned}
S_n(X) &:= S_n(X(t_0), \dots, X(t_n)) \\
&= i \sum_{j=1}^n A\left(\frac{X(t_{j-1}) + X(t_j)}{2}\right) \cdot (X(t_j) - X(t_{j-1})) + \sum_{j=1}^n V\left(\frac{X(t_{j-1}) + X(t_j)}{2}\right) \frac{t}{n}
\end{aligned}$$

Then

$$\begin{aligned}
(T(t/n)^n g)(x) &= \overbrace{\int_{\mathbf{R}^d} \dots \int_{\mathbf{R}^d}}^{n \text{ times}} \prod_{j=1}^n k_0(t/n, x_{j-1} - x_j) e^{-S_n(x_0, \dots, x_n)} g(x_n) dx_1 \dots dx_n \\
&= \int_{D_x} e^{-S_n(X)} g(X(t)) d\lambda_x(X).
\end{aligned}$$

We can show

Proposition 5.2. $T(t/n)^n g \rightarrow e^{-t[H^{(1)}-m]} g$ in $L^2(\mathbf{R}^d)$, $n \rightarrow \infty$.

Proof is omitted.

Now we are in a position to complete the proof of Theorem 5.1. By Proposition 5.2, we see the left-hand side of (5.13) converges to $e^{-t[H^{(1)}-m]} g$ as $n \rightarrow \infty$. On the other hand, we see by Itô's formula [see *) below] that the right-hand side converges to $\int_{D_x} e^{-S(X)} g(X(t)) d\lambda_x(X)$ by Lebesgue convergence theorem. \square

*) For instance, in $t_{j-1} \leq s < t_j$, we have by Itô's formula,

$$\begin{aligned}
& A\left(\frac{X(t_{j-1}) + X(t_j)}{2}\right) \cdot (X(t_j) - X(t_{j-1})) \\
&= \int_{t_{j-1}}^{t_j^+} \int_{|y|>0} \left[A\left(\frac{X(s-) + X(t_{j-1}) + yI_{|y|\geq 1}(y)}{2}\right) \cdot (X(s-) - X(t_{j-1}) + yI_{|y|\geq 1}(y)) \right. \\
&\quad \left. - A\left(\frac{X(s-) + X(t_{j-1})}{2}\right) \cdot (X(s-) - X(t_{j-1})) \right] N_X(dsdy) \\
&\quad + \int_{t_{j-1}}^{t_j^+} \int_{|y|>0} \left[A\left(\frac{X(s-) + X(t_{j-1}) + yI_{|y|<1}(y)}{2}\right) \cdot (X(s-) - X(t_{j-1}) + yI_{|y|<1}(y)) \right. \\
&\quad \left. - A\left(\frac{X(s-) + X(t_{j-1})}{2}\right) \cdot (X(s-) - X(t_{j-1})) \right] \tilde{N}(dsdy) \\
&\quad + \int_{t_{j-1}}^{t_j} \int_{|y|>0} \left[A\left(\frac{X(s) + X(t_{j-1}) + yI_{|y|<1}(y)}{2}\right) \cdot (X(s) - X(t_{j-1}) + yI_{|y|<1}(y)) \right. \\
&\quad \left. - A\left(\frac{X(s) + X(t_{j-1})}{2}\right) \cdot (X(s) - X(t_{j-1})) \right. \\
&\quad \left. - I_{|y|<1}(y) \left\{ \left(\frac{1}{2} (y \cdot \nabla) A \right) \left(\frac{X(s) + X(t_{j-1})}{2} \right) \cdot (X(s) - X(t_{j-1})) \right. \right. \\
&\quad \left. \left. + y \cdot A\left(\frac{X(s) + X(t_{j-1})}{2}\right) \right\} \right] ds n(dy).
\end{aligned}$$

(2) Next we come to the case for the pseudo-differential operator modified by Iftimie-Măntoiu-Purice : $H^{(2)} := H_A^{(2)} + V$ in Definition 3.2. By exactly the same argument as used to show (5.5), we can show that

$$\begin{aligned}
([H_A^{(2)} - m]f)(x) &= - \int_{|y|>0} [e^{-iy \cdot \int_0^1 A(x+\theta y) d\theta} f(x+y) - f(x) \\
&\quad - I_{\{|y|<1\}} y \cdot (\nabla - iA(x)) f(x)] n(dy) \\
&= - \lim_{r \downarrow 0} \int_{|y| \geq r} [e^{-iy \cdot \int_0^1 A(x+\theta y) d\theta} f(x+y) - f(x)] n(dy) \\
(5.13) \quad &= - \text{p.v.} \int_{|y|>0} [e^{-iy \cdot \int_0^1 A(x+\theta y) d\theta} f(x+y) - f(x)] n(dy).
\end{aligned}$$

Theorem 5.3. [IfMP07, IfMP08, IfMP10]

$$\begin{aligned}
(e^{-t[H^{(2)} - m]}g)(x) &= \int_{D_x([0, \infty) \rightarrow \mathbb{R}^d)} e^{-S^{(2)}(t, X)} g(X(t)) d\lambda_x(X), \\
S^{(2)}(t, X) &= i \int_0^{t^+} \int_{|y| \geq 1} \left(\int_0^1 A(X(s-) + \theta y) \cdot y d\theta \right) N_X(dsdy) \\
&\quad + i \int_0^{t^+} \int_{0 < |y| < 1} \left(\int_0^1 A(X(s-) + \theta y) \cdot y d\theta \right) \tilde{N}_X(dsdy) \\
(5.14) \quad &\quad + i \int_0^t ds \text{p.v.} \int_{0 < |y| < 1} \left(\int_0^1 A(X(s) + \theta y) \cdot y d\theta \right) n(dy) + \int_0^t V(X(s)) ds.
\end{aligned}$$

The proof of Theorem 5.3 will be done in exactly the same way as that of Theorem 5.1. Indeed, we have only to replace $A(X(s-) + \frac{y}{2}) \cdot y$ by $\int_0^1 A(X(s-) + \theta y) \cdot y d\theta$.

(3) Finally, we consider the case for the operator defined, in Definition 3.3, with the square root of a nonnegative selfadjoint operator, $H^{(3)} := H_A^{(3)} + V$.

On the one hand, we can determine by functional analysis, namely, by theory of fractional powers (e.g. [Y68, Chap.IX,11, pp.259–261]) $e^{-t[H_A^{(3)}-m]}$ from the nonnegative selfadjoint operator $S := (-i\nabla - A(x))^2 + m^2 =: 2mH_A^{NR} + m^2$ where H_A^{NR} stands for the magnetic nonrelativistic Schrödinger operator $\frac{1}{2m}(-i\nabla - A(x))^2$ without scalar potential. Indeed, we have

$$(5.15) \quad \begin{aligned} e^{-t[H_A^{(3)}-m]}g &= \begin{cases} e^{mt} \int_0^\infty f_t(\lambda) e^{-\lambda S} g d\lambda, & t > 0, \\ 0, & t = 0 \end{cases} \\ f_t(\lambda) &= \begin{cases} (2\pi i)^{-1} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{z\lambda - tz^{1/2}} dz, & \lambda \geq 0, \\ 0, & \lambda < 0 \quad (\sigma > 0). \end{cases} \end{aligned}$$

Here we quickly insert the *Feynman–Kac–Itô formula* (e.g. [S05]) for the magnetic nonrelativistic Schrödinger operator $H^{NR} := H_A^{NR} + V := \frac{1}{2m}(-i\nabla - A(x))^2 + V(x)$ ($m > 0$), a more general formula than the Feynman–Kac formula (2.11):

$$(5.16) \quad \begin{aligned} (e^{-tH^{NR}}g)(x) &= \int_{C_x([0,\infty) \rightarrow \mathbf{R}^d)} e^{-[i \int_0^t A(B(s))dB(s) + \frac{i}{2} \int_0^t \operatorname{div} A(B(s))ds + \int_0^t V(B(s))ds]} g(B(t)) d\mu_x(B) \\ &\equiv \int_{C_x([0,\infty) \rightarrow \mathbf{R}^d)} e^{-[i \int_0^t A(B(s)) \circ dB(s) + \int_0^t V(B(s))ds]} g(B(t)) d\mu_x(B). \end{aligned}$$

This can provide a kind of path integral representation for $e^{-t[H_A^{(3)}-m]}g$ with the Wiener measure, by substituting the Feynman–Kac–Itô formula (5.17) for $V = 0$ with $t = 2m\lambda$ into $e^{-\lambda(S-m^2)} = e^{-2m\lambda H_A^{NR}}$ in the integrand of equation (5.16) for $e^{-t[H_A^{(3)}-m]}g$. Then, to represent $e^{-t[H^{(3)}-m]}g$ for $V \neq 0$, we might apply the Trotter–Kato product formula

$$(5.17) \quad e^{-t[H^{(3)}-m]} = s\text{-}\lim_{n \rightarrow \infty} \left(e^{-(t/n)[H_A^{(3)}-m]} e^{-(t/n)V} \right)^n,$$

to the sum $H^{(3)} - m = (H_A^{(3)} - m) + V$ to express the semigroup $e^{-t[H^{(3)}-m]}$ as a “limit”, where convergence of the right-hand side usually takes place in strong sense as indicated, but now even, *in operator norm*, by the recent results on operator norm convergence [IT01], [ITTZ01] (cf. [IT04], [IT06]). However it is not clear whether this procedure could further yield a path integral representation for $e^{-t[H^{(3)}-m]}g$.

On the other hand, it does not seem possible to represent $e^{-t[H^{(3)}-m]}g$ by path integral through directly applying Lévy process, as we saw in the cases for $e^{-t(H^{(1)}-m)}g$ and $e^{-t(H^{(2)}-m)}g$, because $H_A^{(3)}$ does not seem to be explicitly expressed by a pseudo-differential operator of a certain tractable symbol. It was in this situation that the problem of path integral representation for $e^{-t[H^{(3)}-m]}g$ was studied first by DeAngelis–Serva and Rinaldi [AnSe90, AnRSe91] with use of *subordination/time-change* of Brownian motion, and recently more extensively in [HIL09]

not only for the magnetic relativistic Schrödinger operator $H_A^{(3)}$ but also for Bernstein functions of the magnetic nonrelativistic Schrödinger operator even with spin. To proceed, let us explain about subordination.

Let $B^1(t)$ be the one-dimensional standard Brownian motion being a function in $C_0([0, \infty) \rightarrow \mathbf{R})$ with $B^1(0) = 0$, so that $e^{-t\frac{\xi^2}{2}} = \int_{C_0([0, \infty) \rightarrow \mathbf{R})} e^{i\xi B^1(t)} d\mu_0^1(B^1)$ with μ_0^1 the Wiener measure on $C_0([0, \infty) \rightarrow \mathbf{R})$. Put

$$(5.18) \quad T(t) := \inf\{s > 0; B^1(s) + \sqrt{m}s = \sqrt{m}t\}, \quad t \geq 0.$$

Then $T(t)$ is a monotone, non-decreasing function on $[0, \infty)$ with $T(0) = 0$, belonging to $D_0([0, \infty) \rightarrow \mathbf{R})$ and so becoming a one-dimensional Lévy process, called *subordinator*. Let ν_0 be the associated probability measure on $D_0([0, \infty) \rightarrow \mathbf{R})$.

Proposition 5.4. (e.g. [Ap09, p.54, Example 1.3.21])

$$(5.19) \quad e^{-t[\sqrt{2m\sigma+m^2}-m]} = \int_{D_0([0, \infty) \rightarrow \mathbf{R})} e^{-T(t)\sigma} d\nu_0(T), \quad \sigma \geq 0.$$

This proposition implies that the chacteristic function of the measure ν_0 is given by

$$\begin{aligned} e^{-t\phi(\rho)} &= \int_{D_0([0, \infty) \rightarrow \mathbf{R})} e^{iT(t)\rho} d\nu_0(T), \quad \rho \in \mathbf{R}, \\ \phi(\rho) &= \left(\frac{m}{2}\right)^{1/2} \frac{\sqrt{m^2 + \rho^2} - m}{(\sqrt{m^2 + \rho^2} + m)^{1/2} + \sqrt{2m}^{1/2}} - \frac{(2m)^{1/2}\rho}{(\sqrt{m^2 + \rho^2} + m)^{1/2}} i. \end{aligned}$$

To see this, first analytically extend $\sqrt{2m\sigma+m^2}$ to the right-half complex plane $z := \sigma + i\rho$, $\sigma > 0, \rho \in \mathbf{R}$, and then we have $\phi(\rho) = \lim_{\sigma \rightarrow +0} \sqrt{2m(\sigma + i\rho) + m^2} - m$, of which the right-hand side is calculated as above.

We are in a position to give a path integral representation for $e^{-t[H^{(3)}-m]}g$.

Theorem 5.5. [AnSe90, AnRSe91; HIL09]

$$\begin{aligned} (e^{-t[H^{(3)}-m]}g)(x) &= \int \int_{\substack{C_x([0, \infty) \rightarrow \mathbf{R}^d) \\ \times D_0([0, \infty) \rightarrow \mathbf{R})}} e^{-S^{(3)}(t, B, T)} g(B(T(t))) d\mu_x(B) d\nu_0(T), \\ S^{(3)}(t, B, T) &= i \int_0^{T(t)} A(B(s)) dB(s) + \frac{i}{2} \int_0^{T(t)} \operatorname{div} A(B(s)) ds + \int_0^t V(B(T(s))) ds, \\ (5.20) \quad &\equiv i \int_0^{T(t)} A(B(s)) \circ dB(s) + \int_0^t V(B(T(s))) ds, \end{aligned}$$

where μ_x is the Wiener measure on $C_x([0, \infty) \rightarrow \mathbf{R}^d)$.

Proof of Theorem 5.5. (Sketch) We use Proposition 5.4 and the Feynman–Kac–Itô formula (5.17). Note that $H_A^{(3)} = \sqrt{2mH_A^{NR} + m^2}$. By Spectral Theorem for the nonnegative selfadjoint

operator H_A^{NR} , we have $H_A^{NR} = \int_{\text{Spec}(H_A^{NR})} \sigma dE(\sigma)$. Then for $f, g \in L^2(\mathbf{R}^d)$

$$\langle f, e^{-t[H_A^{(3)} - m]} g \rangle = \int_{\text{Spec}(H_A^{NR})} e^{-t[\sqrt{2m\sigma + m^2} - m]} \langle f, dE(\sigma) g \rangle,$$

where $\langle \cdot, \cdot \rangle$ stands for the inner product of the Hilbert space $L^2(\mathbf{R}^2)$. By Proposition 5.4 and again by Spectral Theorem,

$$\begin{aligned} \langle f, e^{-t[H_A^{(3)} - m]} g \rangle &= \int_{\text{Spec}(H_A^{NR})} \int_{D_0([0, \infty) \rightarrow \mathbf{R})} e^{-T(t)\sigma} d\nu_0(T) \langle f, dE(\sigma) g \rangle \\ &= \int_{D_0([0, \infty) \rightarrow \mathbf{R})} \langle f, e^{-T(t)H_A^{NR}} g \rangle d\nu_0(T). \end{aligned}$$

Applying the Feynman–Kac–Itô formula (5.17) (with $V = 0$) to $e^{-T(t)H_A^{NR}} g$ on the right, we have

$$\begin{aligned} &\langle f, e^{-t[H_A^{(3)} - m]} g \rangle \\ &= \int_{D_0([0, \infty) \rightarrow \mathbf{R})} d\nu_0(T) \int_{\mathbf{R}^d} dx \overline{f(B(0))} \int_{C_x([0, \infty) \rightarrow \mathbf{R}^d)} e^{-i \int_0^{T(t)} A(B(s)) \circ dB(s)} g(B(T(t))) d\mu_x(B) \\ &= \int_{\mathbf{R}^d} dx \overline{f(x)} \int_{\substack{C_x([0, \infty) \rightarrow \mathbf{R}^d \\ \times D_0([0, \infty) \rightarrow \mathbf{R})}} e^{-i \int_0^{T(t)} A(B(s)) \circ dB(s)} g(B(T(t))) d\mu_x(B) d\nu_0(T), \end{aligned}$$

where note $B(0) = x$. This proves the assertion when $V = 0$.

When $V \neq 0$, with partition of $[0, t]$: $0 = t_0 < t_1 < \dots < t_n = t$, $t_j - t_{j-1} = t/n$, we can express $e^{-t[H_A^{(3)} - m]} g = e^{-t[H_A^{(3)} - m] + V}$ by the Trotter–Kato formula (5.18). Rewrite the product of these n operators by path integral with respect to the product of two probability measures $\nu_0(T) \cdot \mu_x(B)$ and note that $T(0) = T(t_0) = 0$, $B(0) = B(T(t_0)) = x$, then we have

$$\begin{aligned} &\langle f, (e^{-(t/n)[H_A^{(3)} - m]} e^{-(t/n)V})^n g \rangle \\ &= \int_{\mathbf{R}^d} dx \int_{D_0([0, \infty) \rightarrow \mathbf{R})} d\nu_0(T) \int_{C_x([0, \infty) \rightarrow \mathbf{R}^d)} \overline{f(B(0))} \\ &\quad \times e^{-i \sum_{j=1}^n \int_{T(t_{j-1})}^{T(t_j)} A(B(s)) \circ dB(s)} e^{-\sum_{j=1}^n V(B(T(t_j))) \frac{t}{n}} g(B(t_n)) d\mu_x(B). \end{aligned}$$

We see, as $n \rightarrow \infty$, that the left-hand side converges to $\langle f, e^{-t[H_A^{(3)} - m]} g \rangle$, and the right-hand side also converges to the goal formula by the Lebesgue theorem, as integral by $dx \cdot \nu_0(T) \cdot \mu_x(B)$. Hence or similarly we can also get (5.21). \square

Finally, as summary, we will collect the three path integral representation formulas in Theorems 5.1, 5.3, 5.5, below, so as to be able to easily see x -dependence. To do so, make change of space, probability measure and paths by translation:

$D_x \rightarrow D_0, \lambda_x \rightarrow \lambda_0, X(s) \rightarrow X(s) + x, B(s) \rightarrow B(s) + x, B(T(s)) \rightarrow B(T(s)) + x$, then

$$(5.10): \quad (e^{-t[H^{(1)}-m]}g)(x) = \int_{D_0([0,\infty)\rightarrow\mathbf{R}^d)} e^{-S^{(1)}(t,X)} g(X(t)+x) d\lambda_0(X),$$

$$S^{(1)}(t,X) = i \int_0^{t+} \int_{|y|\geq 1} A(X(s-)+x+\frac{y}{2}) \cdot y N_X(dsdy) \\ + i \int_0^{t+} \int_{0<|y|<1} A(X(s-)+x+\frac{y}{2}) \cdot y \tilde{N}_X(dsdy) \\ + i \int_0^t ds \text{ p.v. } \int_{0<|y|<1} A(X(s)+x+\frac{y}{2}) \cdot y n(dy) \\ + \int_0^t V(X(s)+x) ds;$$

$$(5.15): \quad (e^{-t[H^{(2)}-m]}g)(x) = \int_{D_0([0,\infty)\rightarrow\mathbf{R}^d)} e^{-S^{(2)}(t,X)} g(X(t)+x) d\lambda_0(X),$$

$$S^{(2)}(t,X) = i \int_0^{t+} \int_{|y|\geq 1} \left(\int_0^1 A(X(s-)+x+\theta y) \cdot y d\theta \right) N_X(dsdy) \\ + i \int_0^{t+} \int_{0<|y|<1} \left(\int_0^1 A(X(s-)+x+\theta y) \cdot y d\theta \right) \tilde{N}_X(dsdy) \\ + i \int_0^t ds \text{ p.v. } \int_{0<|y|<1} \left(\int_0^1 A(X(s)+x+\theta y) \cdot y d\theta \right) n(dy) \\ + \int_0^t V(X(s)+x) ds;$$

$$(5.21): \quad (e^{-t[H^{(3)}-m]}g)(x) = \iint_{\substack{C_0([0,\infty)\rightarrow\mathbf{R}^d) \\ \times D_0([0,\infty)\rightarrow\mathbf{R})}} e^{-S^{(3)}(t,B,T)} g(B(T(t))+x) d\mu_0(B) d\nu_0(T),$$

$$S^{(3)}(t,B,T) = i \int_0^{T(t)} A(B(s)+x) \cdot dB(s) + \frac{i}{2} \int_0^{T(t)} \text{div} A(B(s)+x) ds \\ + \int_0^t V(B(T(s))+x) ds, \\ \equiv i \int_0^{T(t)} A(B(s)+x) \circ dB(s) + \int_0^t V(B(T(s))+x) ds$$

§ 6. Feynman and Dirac

Finally, I would like to close these notes to write something about Feynman and Dirac. In §1, we observed Feynman's Two Postulates equivalent to "path integral". In them, equation (2.5) saying that $\varphi[X]$ is "proportional to" $e^{S(X)/\hbar}$ is the pivotal point. As he himself wrote in his celebrated paper [F48], "this formulation was suggested by some of Dirac's remarks

([D33, D35], [D45])”, though. For what Dirac had criptically remarked as “analogous to” there, Feynman believed to be able to substitute “proportional to” (see Preface of [FH65]).

There has recently been published a book entitled *The Strangest Man: The Hidden Life of Paul Dirac, Quantum Genius*, by Graham Farmelo [Faber and Faber Ltd, London, 2009; paperback ed. 2010]. This volume describes in detail the life of Dirac from his birth to death with much favor and affection. From it I have learned something novel which lets me think again about how it was when Feynman had met Dirac, and how Feynman had been thinking afterwards.

*Time: September 1946

*Place: Conference on ‘The Future of Nuclear Science’, Princeton’s Graduate College.

Feynman was Chairman to introduce Dirac to the audience. The following 12 lines are cited from this book by Graham Farmelo, Chap. 24, p. 333.

Feynman described in his problem to Dirac and came to crunch:

FEYNMAN: Did you know that they were propotional ?

DIRAC: Are they ?

FEYNMAN: Yes they are.

DIRAC: That’s interesting.

Dirac then got up and walked away. Feynman subsequently became famous for his version of quantum mechanics but thought the credit was undeserved. The more closely he looked at the ‘little paper’, the more he realized that he had done nothing new. He later said, repeatedly, ‘I don’t know what all the fuss is about — Dirac did it all before me.’

[Interview with Freeman Dyson, 27 June 2005. Dyson noted that Feynman made the point repeatedly.]

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